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SOMMERFELD-TYPE CONDITIONS AND UNIQUENESS OF THE SOLUTION OF EXTERIOR PROBLEMS OF THE THEORY OF ELASTIC VIBRATIONS OF ANISOTROPIC MEDIA

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Steady vibrations of unbounded elastic anisotropic media are considered in three dimensions in the general case of 21 elastic constants. The problem is posed of extracting a single solution, defined in the whole space, of a system of elliptic equations for the stationary part of the displacement. The asymptotic of the fundamental solution is investigated at infinity and radiation conditions are determined which go directly over into the Sommerfeld conditions when going over to isotropic media. A uniqueness theorem is formulated for the inhomogeneous problem in unbounded domains.

1. The investigation of steady vibrations of elastic anisotropic media in unbounded domains reduces to investigating a system of elliptic equations in the whole space R^n $(n = 3)$ for the stationary part of the displacement. While correct formulations of the boundary value problems for elliptic equations in bounded regions has been studied well, correct formulations of the boundary value problem for equations in unbounded domains are known to a very much lesser degree. This is related to the fact that in addition to conditions on the domain boundary, it is still necessary to give some conditions at infinity, where until recently, there was no answer to this question for the majority of types of hypoelliptic equations.

The system of equations for the stationary part of the displacements in an elastic anisotropic medium belongs to the type of equations whose characteristic polynomial $P(\sigma)$ satisfies the following conditions:

1°. $P(\sigma)$ has only real coefficients.

2°. $P(\sigma)$ has real zeroes.

We assume that the following condition is also satisfied (regular case).

3°. grad $P(\sigma) \neq 0$ at the real zeroes of $P(\sigma)$.

It follows from the conditions listed that the real zeroes of the polynomial form several closed surfaces S_l (ovals).

The Helmholtz equation, for which Sommerfeld obtained the conditions at infinity (the Sommerfeld radiation conditions), belongs to the type of equations mentioned. In the case of isotropic media, the surfaces of the real zeroes of the characteristic equation possess spherical symmetry, and the Sommerfeld conditions are the conditions at infinity.

Hypoelliptic operators of a general kind satisfying the conditions 1° — 3° have recently been considered for the case $n = 2$ [1] and under an additional condition for $n \geqslant 2$.

4°. The total curvature of the surface of real zeroes S_l is different from zero everywhere (the case of strictly convex surfaces S_l) [1, 2].

The hypoelliptic equations examined in [2] are written in the form

$$P(D)\,u(x) = f(x) \qquad (1.1)$$
$$(D = i^{-1}(\partial / \partial x_1, \ldots, \partial / \partial x_n),\ x = (x_1, \ldots, x_n))$$

where $P(\sigma)$ satisfies the conditions $1° - 4°$.

The point $(\sigma_1, \ldots, \sigma_n)$ is denoted by σ.

The main result of this paper is that if a hypoelliptic polynomial $P(\sigma)$ satisfies conditions $1° - 4°$ and the conditions

$$u(x) - o(r^{-(n-3)/2}),\ Q(\omega, D)\,u(x) = o(r^{-(n-1)/2}) \qquad (1.2)$$

are satisfied at infinity, then the solution of the homogeneous equation

$$P(D)u(x) = 0 \qquad (1.3)$$

must be identically zero.

Here $Q(\omega, D)$ is a certain differential operator whose coefficients depend only on ω (ω is the unit vector in the space x), which satisfies the following conditions at points on the surfaces S_l

$$Q[\omega, \sigma_{\pm}^l(\omega)] = 0,\ Q[\omega, \sigma_{\mp}^l(\omega)] \neq 0,\ 1 \leqslant l \leqslant m \qquad (1.4)$$

Here the upper and lower signs in the subscripts are selected depending on S_l, where $\sigma_+^l(\omega)$ ($\sigma_-^l(\omega)$) denotes a point on S_l at which the normal to S_l agrees with (is opposite to) the direction from ω. It is shown in [2] that the Sommerfeld conditions have precisely such a nature. In the general case there is great arbitrariness in the selection of the operator $Q(\omega, \sigma)$. But it turns out that two polynomials Q_1 and Q_2 which vanish simultaneously, or are simultaneously different from zero at the points σ_+^l and σ_-^l, extract the same solution of (1.1) by using conditions (1.2) and (1.4).

The main purpose of this paper is to obtain specific conditions at infinity which will go over into the Sommerfeld radiation conditions in the limit case of isotropic media, and the formulation of a uniqueness theorem based on them. It will be shown that the conditions at infinity have the same nature as do (1.2) and (1.4).

2. The equations of motion of elastic anisotropic media in the general case of 21 elastic constant can be written in the form

$$c_{pqrs}\frac{\partial^2 u_r}{\partial x_q \partial x_s} = \frac{\partial^2 u_p}{\partial t^2} \qquad (2.1)$$

The dependent variables u_p $(p = 1, 2, 3)$ are Cartesian components of the elastic displacement vector, x_q are Cartesian coordinates, t is the time, and c_{pqrs} denotes the elastic constants a_{pqrs} divided by the density ρ.

The summation over q, r, s is in conformity with the rules of tensor calculus. The constants a_{pqrs} are bounded by conditions of positive definiteness of the elastic energy (ε_{pq} are the strain components)

$$2F = a_{pqrs}\varepsilon_{pq}\varepsilon_{rs}, \quad \varepsilon_{pq} = \varepsilon_{qp}$$

Let us consider the plane wave

$$u_p = A_p \varphi (\theta \cdot x - t) \tag{2.2}$$

Substituting into (2.1), we obtain the solvability condition for the homogeneous system (2.1) in the form

$$\det [v^2 \delta_{pr} - c_{pqrs} \eta_q \eta_s] = 0 \tag{2.3}$$

Here $\theta_r = \eta_r |\theta|$, $|\eta| = 1$, where $v = |\theta|^{-1}$ is the phase velocity of plane wave propagation.

The expression on the left in (2.3) is a characteristic polynomial of the system (2.1) of sixth order in θ_q. Its real roots form three closed surfaces Σ_l $(l = 1, 2, 3)$ in the space θ_1, θ_2, θ_3, which are branches of a sixth order algebraic surface, and are the geometric locus of the ends of vectors of the length v^{-1} drawn from the origin. These surfaces are called slowness surfaces because of the inverse dependence of the vector length on the velocity.

Let us number them so that $v_1^2(\eta) > v_2^2(\eta) \geqslant v_3^2(\eta)$. In the regular case, there is no equality sign, and the separate branches Σ_l (ovals) of the surface Σ have no common points.

If all the ovals Σ_l are convex, then the wave surfaces are also convex. The outer oval of the wave surface corresponds to the inner oval Σ_1. Since Σ_1 is always convex, the outer wave front is also always convex. The remaining wave fronts can have a complex shape and contain acute-angled edges. Within a domain bounded by the outer wave front there are hence observed domains where the fundamental solution vanishes identically, this is the so-called lacuna. In the spatial case $n = 3$ one of the lacunae is a domain bounded by the inner wave front. The fundamental solution is also identically zero in the part of space in front of the outer wave front, and is also a lacuna in this sense.

I. G. Petrovskii [3] established the necessary and sufficient conditions for the existence of lacunae. These conditions depend on the topological properties of the section of the surface $P(1, \theta_1, \ldots, \theta_n) = 0$ corresponding to the linear hyperbolic equation

$$P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) u(x) = 0$$

executed by the plane $\theta \cdot x + t = 0$. The confirmation of these conditions, formulated in terms of the homology of definite cycles on the surface $P(1, \theta_1, \ldots, \theta_n)$ to zero, is an independent complex problem for the system (2.1) with $n = 3$. The problem is simplified radically for $n = 2$ as well as for $(n = 3)$ in the particlar case when all the surfaces Σ_l are convex. The lacunae for convex Σ_l with $n = 3$ are just a domain bounded by the inner wave front, and the domain in front of the outer front, and the domains between the fronts cannot be lacunae [4].

3. We shall be interested in solutions of the form

$$u_p = u_{p0}(x) \exp(-i\alpha t) \tag{3.1}$$

We obtain an elliptic system of equations

$$c_{pqrs}\frac{\partial^2 u_{r0}}{\partial x_q \partial x_s} + \alpha^2 u_{p0} = 0 \tag{3.2}$$

from (1.1) for the stationary part of the displacements u_{p0}.

We shall henceforth consider just the stationary part of the solution, where the zero subscript will be omitted.

Let us consider a system of the form (3.2) with a right side $f_p(x)$ which is a certain finite function of the coordinates

$$c_{pqrs}\frac{\partial^2 u_r}{\partial x_q \partial x_s} + \alpha^2 u_p = f_p \tag{3.3}$$

and let us be interested in solutions of the system (3.3) defined in the whole space R^n, $n = 3$. From the physical viewpoint the right side $f_p(x)$ of the system (3.3) defines a system of distributed vibrations sources, given in a finite part of space. This system of sources should naturally define the displacement field uniquely in the whole space, where the displacements should tend to zero at infinity. At the same time it is known that the system of equations (3.3) of the Helmholtz equations type allows an infinite set of solutions if we limit ourselves just to the requirement that the solution decrease at infinity. This is related to the presence of an infinite set of solutions of the homogeneous system (3.2) decreasing at infinity.

Before determining the specific form of the condition at infinity such that the solutions of the homogeneous problem which satisfy them can only be identically zero, it is first necessary to study the asymptotic of the fundamental matrix of the system (3.2) at infinity, i.e., solutions of a system with a right side in the form of a delta function of the coordinate

$$c_{pqrs}\frac{\partial^2 u_r}{\partial x_q \partial x_s} + \alpha^2 u_p = -\delta_p^{\ j}\delta(x) \tag{3.4}$$

Applying a generalized Fourier transform to both sides of (3.4), solving the system obtained for the Fourier transform, and then applying the inverse Fourier transform, we obtain the solution of the problem in the form

$$u_p^{\ j} = (2\pi)^{-3}\int_H W_p^{\ j}(s)\frac{\exp[i(s\cdot x)]}{P(s)}\,ds \tag{3.5}$$

Here H is the appropriate Hermander ladder, which is a discontinuous set in four-dimensional space defined by real σ_1, σ_2, σ_3 and the imaginary component τ_1. Its involvement is related to the need to emerge in the complex domain of at least one variable for the existence of the integral [1]. We take a hyperplane as H such that for fixed σ_2, σ_3 $(-\infty < \sigma_2 < \infty,\ -\infty < \sigma_3 < \infty)$ the appropriate section of H is a line in the plane $s = \sigma_1 + i\tau_1$ parallel to the real σ_1 axis.

The functions $U_p^{\ j} = W_p^{\ j}(s)\ /\ P(s)$ are determined from the solution of the system

$$(L - \alpha^2 E)\,U^j = A^j, \quad A^j = (\delta_1^{\ j},\ \delta_2^{\ j},\ \delta_3^{\ j})$$
$$U^j = (U_1^{\ j},\ U_2^{\ j},\ U_3^{\ j}), \quad L = \|\sigma_q\sigma_s c_{pqrs}\|$$

Here $P(s)$ is the characteristic polynomial, $W_p{}^j = \det M^j$, where M^j is the matrix being obtained upon replacement of the p column of the matrix $L - \alpha^2 E$ by the vecotor A^j, E is the unit matrix, and $\delta_p{}^j$ is the Kronecker symbol.

4. Let us consider an arbitrary direction in space x defined by the unit vector $\omega = (\omega_1, \omega_2, \omega_3)$, and we select the coordinates so that the x_1 axis would be in the direction of the vector ω. We rewrite (3.5) in the form

$$u_p{}^j = (2\pi)^{-3} \iint\limits_{-\infty}^{\infty} d\sigma_2 d\sigma_3 \left(\int\limits_{-\infty}^{\infty} W_p{}^j(\sigma') \frac{\exp(is_1x_1)}{P(\sigma')} d\sigma_1 \right) \tag{4.1}$$

$$(\sigma' = (s_1, \sigma_2, \sigma_3), \ s_1 = \sigma_1 + i\tau_1)$$

The denominator of the integrand vanishes on the surface S (consisting of three separate branches of S_l). Fixing σ_2, σ_3 and integrating with respect to σ_1 we obtain the following expression by using residue theory:

$$u_p{}^j = \frac{i}{8\pi^2} \iint\limits_{-\infty}^{\infty} W_p{}^j(\sigma) \frac{\exp(i\sigma_1 x_1)}{P'_{\sigma 1}(\sigma)} d\sigma_2 d\sigma_3 \tag{4.2}$$

where $\sigma_1, \sigma_2, \sigma_3$ are related by the equation $P(\sigma) = 0$, but $P_{\sigma 1}'(\sigma) \neq 0$ because of condition 3°, $\sigma 1 \equiv \sigma_1$.

To estimate the asymptotic value of $u_p{}^j$ as $x_1 \to \infty$ we use the stationary phase principle [5, 6]. According to this principle, only points on the surfaces of real zeroes of the characteristic equation in which the normal is parallel to the x_1 axis introduce a contribution to the value of the integral asymptotically as $x_1 \to \infty$. In the neighborhood of each such point S_l can be represented in the form

$$\sigma_1 = \sigma_{1\nu} + {}^1/_2\, k_2 (\sigma_2 - \sigma_{2\nu})^2 + {}^1/_2\, k_3 (\sigma_3 - \sigma_{3\nu})^2 + \ldots$$

Here k_2 and k_3 are the principal curvatures taken positive (negative) in the case when the surface in the neighborhood of the point σ_ν is convex (concave). It is assumed that $k_i \neq 0$. The x_2, x_3 axes are selected to coincide with the directions of the principal curvatures. Letting $u_{p\nu}{}^j$ denote the contribution of the point σ_ν, we obtain by the stationary phase principle [5]

$$u_{p\nu}^j = \frac{i}{8\pi^2} \frac{W_p{}^j(\sigma_\nu)}{P'_{\sigma 1}(\sigma_\nu)} \exp(i\sigma_{1\nu} x_1) \times$$

$$\iint\limits_{-\infty}^{\infty} \exp\left[\left(i\frac{k_2}{2}\right)(\sigma_2 - \sigma_{2\nu})^2 x_1 + \left(i\frac{k_3}{2}\right)(\sigma_3 - \sigma_{3\nu})^2 x_1\right] d\sigma_2 d\sigma_3$$

Using the substitution $\sigma_i - \sigma_{i\nu} = t \exp[i(\pi/4)\operatorname{sgn} q_i]$, we have

$$\int\limits_{-\infty}^{\infty} \exp\left[i\left(\frac{k_i}{2}\right)(\sigma_i - \sigma_{i\nu})^2 x_1\right] d\sigma_i = 2C_i \int\limits_{0}^{\infty} \exp(-q_i{}^2 t^2)\, dt = C_i \frac{\sqrt{\pi}}{|q_i|}$$

$$C_i = \exp(i(\pi/4)\operatorname{sgn} q_i), \ q_i = k_i x_1, \ x_1 > 0, \ i = 2,3$$

We finally obtain

$$u_{p\nu}^{j} = \frac{i}{4\pi x_1} \frac{W_p^{\,j}(\sigma_\nu)\exp(i\sigma_{1\nu}x_1)}{\sqrt{|k_2k_3|}\,P'_{\sigma 1}(\sigma_\nu)} \exp\left[i\left(\frac{\pi}{4}\right)(\operatorname{sgn} k_2 + \operatorname{sgn} k_3)\right]$$

In order to obtain an asymptotic expression in the direction of the unit vector ω, it is sufficient to replace x_1 by r, the derivative with respect to σ_1 by the derivative of $P(\sigma)$ with respect to the direction ω, $\sigma_{1\nu}x_1$ by the scalar product $\sigma_\nu \cdot x$ of the vectors $(\sigma_{1\nu}, \sigma_{2\nu}, \sigma_{3\nu})$ and (x_1, x_2, x_3), the product k_2k_3 by the Gaussian curvature k in the initial coordinate system. Taking into account that at the point at which the derivative of $P(\sigma)$ is taken, the normal is parallel to ω, we obtain

$$u_p^{\,j} = \sum_\nu \frac{1}{4\pi r} \frac{KW_p^{\,j}(\sigma_\nu)\exp[ir(\sigma_\nu \cdot \omega)]}{|\operatorname{grad} P(\sigma_\nu)|\sqrt{|k|}} + O(r^{-2}) \tag{4.3}$$

The summation is over all points σ_ν of the surfaces S_l where the normal is parallel to the vector ω. It is assumed that the Gaussian curvature $k \neq 0$ at all these points $K = \pm i$ for $k < 0$ and $\operatorname{grad} P(\sigma)$ is taken in the directions $\pm\omega$, $K = \pm 1$ for $k > 0$ (the surface is convex) relative to $\pm\operatorname{grad} P$. (The surfaces S_l are surfaces of real zeroes in the space of wave numbers σ_i (the vectors σ). The surfaces Σ_l considered above are also surfaces of real zeroes of the characteristic polynomial, but in the space of quantities inverse to the velocities—θ_i).

The expression $\sigma_\nu \cdot \omega$ is the projection of the wave vector $(\sigma_{1\nu}, \sigma_{2\nu}, \sigma_{3\nu})$ drawn from the origin to a point on the surface S where the normal is parallel to the unit vector ω in the direction of the vector ω. The wave vector in an elastic anisotropic medium is a homogeneous function of the first degree in the frequency α, therefore we can write

$$(\sigma_\nu \cdot \omega) = \alpha c_\nu(\omega). \tag{4.4}$$

Here $c_\nu(\omega)$ is a quantity inverse to the ray velocity which is defined as the velocity at which a perturbation occurring at time $t = 0$ at the point $x = 0$ (from a concentrated pulse source, say) reaches the point (x_1, x_2, x_3). In other words, the geometric locus of points which perturbations from a concentrated pulse source will reach at time $t = 1$, will be wave surfaces. We henceforth use the notation

$$k_\nu(\omega) = \alpha c_\nu(\omega) \tag{4.5}$$

Substituting (4.5) into (4.3) and taking account of the factor $\exp(-i\alpha t)$, we have as $r \to \infty$

$$u_p^{\,j} = r^{-1}\sum_\nu T_\nu^{pj}(\omega)\exp\{ik_\nu(\omega)[r - w_\nu(\omega)t]\} + O(r^{-2}) \tag{4.6}$$

where $w_\nu = c_\nu^{-1}$ is the ray velocity in the direction of the vector ω.

We see that the fundamental solution degenerates into a traveling divergent wave at infinity, where the isophase curves coincide with the wave fronts being propagated from the concentrated pulse source. A point with a given phase moves in the direction of the vector ω with the ray velocity. In the case of strictly convex surfaces of real zeroes of the characteristic polynomial, the asymptotic expansion obtained is valid for any ω. The total curvature at any point is positive. In the general case, surfaces of real zeroes can be convex or concave, depending on ω, so that points (or entire curves) are observed on these surfaces where the curvature will vanish. The directions of the normals at these points determine the directions in physical space of the rays which pass through the angular points (reentrant points) on the fronts (wave surfaces). The discussion presented above is valid only for rays which do not pass through the angular points of the lacunae.

Let us examine the case when one of the principal curvatures is zero. In particular this case includes all transversally-isotropic media when S_l are surfaces of revolution. Directing x_1 along the normal to the surface, and the x_2, x_3 axes so that $k \neq 0,\ k_3 = 0$, we have

$$\sigma_1 = \sigma_{1c} + (k\ /\ 2)\,(\sigma_2 - \sigma_{2c})^2 + \lambda_3(\sigma_3 - \sigma_{3c})^3 + \ldots \tag{4.7}$$

In this direction the asymptotic expansion for $x_1 \to \infty$ has the form [5]

$$u_{pc}^j = Q_p^j\,(\sigma_c) \sqrt{\frac{2\pi}{x_1\,|\,k_2\,|}}\,\frac{\sqrt{3}}{3!\,(x_1\lambda_3)^{1/3}}\exp\left(i\sigma_{1c}x_1 + \left(\frac{\pi i}{4}\right)\operatorname{sgn} k_2\right)$$

and, therefore, the wave amplitude decreases as $r^{-5/6}$ in the direction of the rim of an acute-angled edge on the wave surface. The component $(k_3\ /\ 2)\,(\sigma_3 - \sigma_{3c})^2$ should be retained in (4.7) to investigate the asymptotic in the neighborhood of these directions.

5. Taking account of (4.5), we write (4.4) in the form

$$i\,(\sigma_\nu\cdot\omega) - ik_\nu(\omega) = 0 \tag{5.1}$$

The point $\sigma_\nu = (\sigma_{1\nu}, \sigma_{2\nu}, \sigma_{3\nu})$ is a point on S_l at which the normal is parallel to the vector ω and directed to the same side, i.e., the point $\sigma_+^l\,(\omega)$. Denoting the left side of (5.1) by $Q_l\,(\omega, \sigma)$, we will have

$$Q\,[\omega,\ \sigma_+^l\,(\omega)] = 0,\ \ Q\,[\omega,\ \sigma_-^l\,(\omega)] \neq 0,\ \ l = 1,\,2,\,3$$

if we take

$$Q\,(\omega,\ \sigma) \equiv \prod_{l=1}^{3}\,[i\,(\sigma\cdot\omega) - ik_l\,(\omega)] \equiv \prod_{l=1}^{3}\,Q_l\,(\omega,\ \sigma)$$

as the polynomial $Q\,(\omega, \sigma)$ considered above.

Replacing the vector σ by the differential vector D, we obtain

$$Q_l\,(\omega,\ D) = \frac{\partial}{\partial r} - ik_l\,(\omega)$$

$$D = i^{-1}\left(\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2},\ \frac{\partial}{\partial x_3}\right),\quad \frac{\partial}{\partial r} = -\,iD\cdot\omega$$

The discussion presented permits formulation of a uniqueness theorem for the case when the surfaces of real zeroes S_l of the characteristic polynomial $P(\sigma)$ are strictly convex.

T h e o r e m 1. Let twice continuously differentiable functions u_p satisfy the system of equations

$$c_{pqrs}\frac{\partial^2 u_r}{\partial x_q \partial x_s} + \alpha^2 u_p = 0 \tag{5.2}$$

in all space $x \in R^n (n = 3)$, and be representable in the form

$$u_p = \sum_{j=1}^{3} u_{pj}$$

$$u_{pj} = O(r^{-1}), \quad \frac{\partial u_{pj}}{\partial r} - ik_j(\omega)\, u_{pj} = o(r^{-1}), \quad r \to \infty \tag{5.3}$$

Then the functions u_p are identically zero.

Isotropic media can be considered as a particular case of media of cubic symmetry under the condition that the elastic constants are connected by the relationship $a_{11} - a_{12} = 2a_{44}$. The surfaces S_l hence become spherical. Two surfaces coincide and we obtain

$$k_2(\omega) = k_3(\omega) = \alpha\sqrt{\frac{\rho}{\mu}}, \quad k_1(\omega) = \alpha\sqrt{\frac{\rho}{\lambda + 2\mu}}$$

where λ, μ are the Lame coefficients. Conditions (5.3) go over into the Sommerfeld conditions. As in the case of the Sommerfeld conditions, we have

$$|u_{pl}| < Cr^{(1-n)/2}, \quad \left|\frac{\partial u_{pl}}{\partial r} - ik_l(\omega)\, u_{pl}\right| < Cr^{-(1+n)/2} \tag{5.4}$$

by combining the writing of the conditions at infinity for the plane $(n = 2)$ [7] and space $(n = 3)$ cases.

In the plane case $p = 1, 2,\ l = p, s$ [7], and in the space case $p = 1, 2, 3,$ $l = 1, 2, 3$. Analogous conditions, but for converging waves, differ from (5.4) just by the sign in front of the factor $ik_l(\omega)$.

In the case of transversally isotropic media, axisymmetric vibrations are allowable, hence the conditions at infinity corresponding to diverging waves are written in the form (5.4) for $n = 3$, where $p = 1, 2,\ l = p, s$ as in the plane case, and the functions $k_l(\omega) = k_l(\varphi)$ do not differ from those presented in [7] for the plane case (φ is an angle measured from the axis of symmetry).

6. The results in Sect. 5 refer to the case of strictly convex surfaces S_l. The investigation presented above of the case when just one of the principal curvatures vanishes already shows the strong difference in behavior of the fundamental solution at infinity in directions coincident with the normal directions at inflection points of the surfaces S_l. Following [2], we shall call the unit vector ω nonsingular it the total curvature is not zero at all points of the surfaces S_l at which the normal is parallel to ω, the total curvature does not equal zero. Let the coefficients of the polynomial $Q(\omega, \sigma)$ be defined only for nonsingular vectors ω, where as before the polynomial $Q(\omega, \sigma)$ vanishes for nonsingular vectors ω at points of S_l at which the normal to S_l is not only parallel to, but also coincides with, ω in direction, and $Q(\omega, \sigma)$ is not zero at points in which the normal direction is opposite to ω.

The nonsingular vectors form an open set on the unit sphere. In the general case when the surfaces S_l satisfy just the first three conditions presented in Sect. 1, it is not possible to solve the problem of studying the asymptotic behavior of fundamental solutions entirely, however, a uniqueness theorem can be formulated fully be using certain integral relations which the solution of the inhomogeneous problem should satisfy.

By using the results in [2], it can be shown that the following is valid:

T h e o r e m 2. If conditions 1° — 3° are satisfied, then for any finite functions $f_p(x)$, the system (3.3) has a solution, which is moreover unique, belongs locally to L_2 and satisfies the conditions

$$\lim_{R\to\infty} R^{-1} \int_{R\leqslant x\leqslant 2R} |u_p(x)|^2 dx < \infty$$

$$\lim_{R\to\infty} R^{-1} \int_{\substack{R\leqslant x\leqslant 2R \\ x\in K}} |Q(\omega, D)\, u_p(x)|^2 dx = 0$$

for any nonsingular cone K with apex at the origin, i.e., a cone whose intersection with the unit sphere consists of nonsingular vectors.

It is here necessary to take

$$Q(\omega, D) = \prod_{l=1}^{N} \left(\frac{\partial}{\partial r} - ik_l(\omega)\right)$$

where the quantity N depends on ω For cones formed by rays which do not intersect lacunae, $N = 3$. For cones comprised of rays which intersect lacunae, the number N is increased by the number of lacuna boundaries intersected by these rays (i.e., at an additional number of points on S_l at which the normals coincide with ω).

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